THEOREMS FROM GRIES AND SCHNEIDER'S LADM

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Abstract. This is a collection of the axioms and theorems in Gries and Schneider's book *A Logical Approach to Discrete Math* (LADM), Springer-Verlag, 1993. The numbering is consistent with that text. Additional theorems not included or numbered in LADM are indicated by a three-part number. This document serves as a reference for homework exercises and taking exams.

Table of Precedences

(a)	$[x := e]$ (textual substitution)	(highest precedence)
(b)	. (function application)	
(c)	unary prefix operators: $+ \quad - \quad \neg \quad \# \quad \sim \quad \mathscr{P}$	
(d)	$**$	
(e)	$\cdot \quad / \quad \div \quad$ **mod** **gcd**	
(f)	$+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$	
(g)	$\downarrow \quad \uparrow$	
(h)	$\#$	
(i)	$\triangleleft \quad \triangleright \quad \wedge$	
(j)	$= \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad \mid$	(conjunctional)
(k)	$\vee \quad \wedge$	
(l)	$\Rightarrow \quad \Leftarrow$	
(m)	$\equiv$	(lowest precedence)

All nonassociative binary infix operators associate from left to right except $**$, $\triangleleft$, and $\Rightarrow$, which associate from right to left.

Definition of $\not$: The operators on lines (j), (l), and (m) may have a slash / through them to denote negation—e.g. $x \notin T$ is an abbreviation for $\neg(x \in T)$.

Some Basic Types

Name	Symbol	Type (set of values)
integer	$\mathbb{Z}$	integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
nat	$\mathbb{N}$	natural numbers: $0, 1, 2, \ldots$
positive	$\mathbb{Z}^+$	positive integers: $1, 2, 3, \ldots$
negative	$\mathbb{Z}^-$	negative integers: $-1, -2, -3, \ldots$
rational	$\mathbb{Q}$	rational numbers: i/j for i, j integers, $j \neq 0$
reals	$\mathbb{R}$	real numbers
positive reals	$\mathbb{R}^+$	positive real numbers
bool	$\mathbb{B}$	booleans: $true, false$

Date: November 21, 2022.

Theorems of the Propositional Calculus

Equivalence and $true$.

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \ \equiv \ (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and $false$.

(3.8) **Definition of $false$:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \ \equiv \ \neg p \equiv q$

(3.10) **Definition of $\not\equiv$:** $(p \not\equiv q) \ \equiv \ \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of $false$:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \ \equiv \ \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \ \equiv \ (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \ \equiv \ (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \ \equiv \ (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \ \ \equiv \ \ p \equiv q \not\equiv r$

(3.19.1) $p \not\equiv p \not\equiv q \ \equiv \ q$

Disjunction.

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction.

(3.35) **Axiom, Golden rule:** $p \wedge q \ \equiv \ p \ \equiv \ q \ \equiv \ p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:**
(a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:**
(a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.46.1) **Consensus:** $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r) \equiv (p \wedge q) \vee (\neg p \wedge r)$

(3.47) **De Morgan:**
(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \ \equiv\ p \wedge \neg q \ \equiv\ \neg p$

(3.49) $p \wedge (q \equiv r) \ \equiv\ p \wedge q \ \equiv\ p \wedge r \ \equiv\ p$

(3.50) $p \wedge (q \equiv p) \ \equiv\ p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \ \equiv\ (p \equiv q) \wedge (r \equiv q)$

(3.52) **Equivalence:** $p \equiv q \ \equiv\ (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \ \equiv\ (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \ \equiv\ p \ \equiv\ q \ \equiv\ r \ \equiv\ p \vee q \ \equiv\ q \vee r \ \equiv\ r \vee p \ \equiv\ p \vee q \vee r$

Implication.

(3.57) **Definition of Implication:** $p \Rightarrow q \ \equiv\ p \vee q \ \equiv\ q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \ \equiv\ q \Rightarrow p$

(3.59) **Implication:** $p \Rightarrow q \ \equiv\ \neg p \vee q$

(3.60) **Implication:** $p \Rightarrow q \ \equiv\ p \wedge q \ \equiv\ p$

(3.61) **Contrapositive:** $p \Rightarrow q \ \equiv\ \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \ \equiv\ p \wedge q \ \equiv\ p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \ \equiv\ (p \Rightarrow q) \ \equiv\ (p \Rightarrow r)$

(3.63.1) **Distributivity of $\Rightarrow$ over $\wedge$:** $p \Rightarrow q \wedge r \ \equiv\ (p \Rightarrow q) \wedge (p \Rightarrow r)$

(3.63.2) **Distributivity of $\Rightarrow$ over $\vee$:** $p \Rightarrow q \vee r \ \equiv\ (p \Rightarrow q) \vee (p \Rightarrow r)$

(3.64) $p \Rightarrow (q \Rightarrow r) \ \equiv\ (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \ \equiv\ p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \ \equiv\ p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \ \equiv\ p$

(3.68) $p \vee (p \Rightarrow q) \ \equiv\ true$

(3.69) $p \vee (q \Rightarrow p) \ \equiv\ q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \ \equiv\ p \ \equiv\ q$

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \ \equiv\ true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \ \equiv\ p$

(3.74) $p \Rightarrow false \equiv \neg p$

(3.74.1) $\neg p \Rightarrow false \equiv p$

(3.74.2) $p \Rightarrow \neg p \equiv \neg p$

(3.75) $false \Rightarrow p \equiv true$

(3.76) **Weakening/strengthening:**

(a) $p \Rightarrow p \vee q$ (Weakening the consequent)

(b) $p \wedge q \Rightarrow p$ (Strengthening the antecedent)

(c) $p \wedge q \Rightarrow p \vee q$ (Weakening/strengthening)

(d) $p \vee (q \wedge r) \Rightarrow p \vee q$

(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.76.1) $p \wedge q \Rightarrow p \vee r$ (Weakening/strengthening)

(3.76.2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.77.1) **Modus tollens:** $(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv p \vee q \Rightarrow r$

(3.78.1) $(p \Rightarrow r) \vee (q \Rightarrow r) \equiv p \wedge q \Rightarrow r$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:**

(a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

(3.82.1) **Transitivity of $\equiv$:** $(p \equiv q) \wedge (q \equiv r) \Rightarrow (p \equiv r)$

(3.82.2) $(p \equiv q) \Rightarrow (p \Rightarrow q)$

Leibniz as an axiom.

This section uses the following notation: E^z_X means $E[z := X]$.

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E^z_e = E^z_f$

(3.84) **Substitution:**

(a) $(e = f) \wedge E^z_e \equiv (e = f) \wedge E^z_f$

(b) $(e = f) \Rightarrow E^z_e \equiv (e = f) \Rightarrow E^z_f$

(c) $q \wedge (e = f) \Rightarrow E^z_e \equiv q \wedge (e = f) \Rightarrow E^z_f$

(3.85) **Replace by $true$:**

(a) $p \Rightarrow E^z_p \equiv p \Rightarrow E^z_{true}$

(b) $q \wedge p \Rightarrow E^z_p \equiv q \wedge p \Rightarrow E^z_{true}$

(3.86) **Replace by $false$:**

(a) $E^z_p \Rightarrow p \equiv E^z_{false} \Rightarrow p$

(b) $E^z_p \Rightarrow p \vee q \equiv E^z_{false} \Rightarrow p \vee q$

(3.87) **Replace by $true$:** $p \wedge E^z_p \equiv p \wedge E^z_{true}$

(3.88) **Replace by $false$:** $p \vee E^z_p \equiv p \vee E^z_{false}$

(3.89) **Shannon:** $E^z_p \equiv (p \wedge E^z_{true}) \vee (\neg p \wedge E^z_{false})$

(3.89.1) $E^z_{true} \wedge E^z_{false} \Rightarrow E^z_p$

Additional theorems concerning implication.

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof technique metatheorems.

(4.4) **Deduction (assume conjuncts of antecedent):**
To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

(4.5) **Case analysis:** If E^z_{true} and E^z_{false} are theorems, then so is E^z_P.

(4.6) **Case analysis:** $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.7.1) **Truth implication:** To prove P, prove $true \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.9.1) **Proof by contradiction:** To prove P, prove $\neg P \equiv false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

General Laws of Quantification

For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$

(8.14) **Axiom, One-point rule:** Provided $\neg occurs(`x', `E')$,
$(\star x \mid x = E : P) = P[x := E]$

(8.15) **Axiom, Distributivity:** Provided $P, Q : \mathbb{B}$ or R is finite,
$(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

(8.16) **Axiom, Range split:** Provided $R \wedge S \equiv false$ and $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.17) **Axiom, Range split:** Provided $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.18) **Axiom, Range split for idempotent $\star$:** Provided $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.19) **Axiom, Interchange of dummies:** Provided $\star$ is idempotent or R and Q are finite,
$\neg occurs(`y', `R'), \neg occurs(`x', `Q')$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

(8.20) **Axiom, nesting:** Provided $\neg occurs(`y', `R')$,
$(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$

(8.21) **Axiom, Dummy renaming:** Provided $\neg occurs(`y', `R, P')$,
$(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$

(8.22) **Change of dummy:** Provided $\neg occurs(`y', `R, P')$, and f has an inverse,
$(\star x \mid R : P) = (\star y \mid R[x := f.y] : P[x := f.y])$

(8.23) **Split off term:** For n: $\mathbb{N}$,

(a) $(\star i \mid 0 \le i < n+1 : P) = (\star i \mid 0 \le i < n : P) \star P[i := n]$

(b) $(\star i \mid 0 \le i < n+1 : P) = P[i := 0] \star (\star i \mid 0 < i < n+1 : P)$

Theorems of the Predicate Calculus

Universal quantification.

Notation: $(\star x \mid : P)$ means $(\star x \mid true : P)$.

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.3) **Trading:**

(a) $(\forall x \mid R : P) \equiv (\forall x \mid : \neg R \vee P)$

(b) $(\forall x \mid R : P) \equiv (\forall x \mid : R \wedge P \equiv R)$

(c) $(\forall x \mid R : P) \equiv (\forall x \mid : R \vee P \equiv P)$

(9.4) **Trading:**

(a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$

(b) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$

(c) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$

(d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$

(9.4.1) **Universal double trading:** $(\forall x \mid R : P) \equiv (\forall x \mid \neg P : \neg R)$

(9.5) **Axiom, Distributivity of $\vee$ over $\forall$:** Provided $\neg occurs(\text{‘}x\text{’}, \text{‘}P\text{’})$,
$P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

(9.6) Provided $\neg occurs(\text{‘}x\text{’}, \text{‘}P\text{’})$, $(\forall x \mid R : P) \equiv P \vee (\forall x \mid : \neg R)$

(9.7) **Distributivity of $\wedge$ over $\forall$:** Provided $\neg occurs(\text{‘}x\text{’}, \text{‘}P\text{’})$,
$\neg(\forall x \mid : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$

(9.8) $(\forall x \mid R : true) \equiv true$

(9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$

(9.10) **Range weakening/strengthening:** $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$

(9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$

(9.12) **Monotonicity of $\forall$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$

(9.13) **Instantiation:** $(\forall x \mid : P) \Rightarrow P[x := E]$

(9.16) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

Existential quantification.

(9.17) **Axiom, Generalized De Morgan:** $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.18) **Generalized De Morgan:**

(a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$

(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$

(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.20) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$

(9.20.1) **Existential double trading:** $(\exists x \mid R : P) \equiv (\exists x \mid P : R)$

(9.20.2) $(\exists x \mid : R) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\exists x \mid R : P))$

(9.21) **Distributivity of $\wedge$ over $\exists$:** Provided $\neg occurs(`x', `P')$,
$P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$

(9.22) Provided $\neg occurs(`x', `P')$, $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid : R)$

(9.23) **Distributivity of $\vee$ over $\exists$:** Provided $\neg occurs(`x', `P')$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$

(9.24) $(\exists x \mid R : false) \equiv false$

(9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$

(9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$

(9.26.1) **Body weakening/strengthening:** $(\exists x \mid R : P \wedge Q) \Rightarrow (\exists x \mid R : P)$

(9.27) **Monotonicity of $\exists$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$

(9.28) **$\exists$-Introduction:** $P[x := E] \Rightarrow (\exists x \mid : P)$

(9.29) **Interchange of quantification:** Provided $\neg occurs(`y', `R')$ and $\neg occurs(`x', `Q')$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.30) Provided $\neg occurs(`\hat{x}', `Q')$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem.

A Theory of Sets

(11.2) **Axiom, Enumeration:** $\{e_0, e_1, \ldots, e_{n-1}\} = \{x \mid x = e_0 \vee x = e_1 \vee \cdots \vee x = e_{n-1} : x\}$

(11.3) **Axiom, Set membership:** Provided $\neg occurs(`x', `F')$,
$F \in \{x \mid R : E\} \equiv (\exists x \mid R : F = E)$

(11.4) **Axiom, Extensionality:** $S = T \equiv (\forall x \mid : x \in S \equiv x \in T)$

(11.4.1) **Axiom, Empty set:** $\emptyset = \{x \mid false : E\}$

(11.4.2) $e \in \emptyset \equiv false$

(11.4.3) **Axiom, Universe:** $\mathbf{U} = \{x \mid : x\}$, $\mathbf{U}$: $set(t) = \{x\!: t \mid : x\}$

(11.4.4) $e \in \mathbf{U} \equiv true$, for e: t and $\mathbf{U}$: $set(t)$

(11.5) $S = \{x \mid x \in S : x\}$

(11.5.1) **Axiom, Abbreviation:** For x a single variable, $\{x \mid R\} = \{x \mid R : x\}$

(11.6) Provided $\neg occurs(`y', `R')$ and $\neg occurs(`y', `E')$,
$\{x \mid R : E\} = \{y \mid (\exists x \mid R : y = E)\}$

(11.7) $x \in \{x \mid R\} \equiv R$
R is the characteristic predicate of the set.

(11.7.1) $y \in \{x \mid R\} \equiv R[x := y]$ for any expression y

(11.9) $\{x \mid Q\} = \{x \mid R\} \equiv (\forall x \mid : Q \equiv R)$

(11.10) $\{x \mid Q\} = \{x \mid R\}$ is valid iff $Q \equiv R$ is valid.

(11.11) **Methods for proving set equality $S = T$:**

(a) Use Leibniz directly.
(b) Use axiom Extensionality (11.4) and prove the (9.8) Lemma
$v \in S \equiv v \in T$ for an arbitrary value v.
(c) Prove $Q \equiv R$ and conclude $\{x \mid Q\} = \{x \mid R\}$.

Operations on sets.

(11.12) **Axiom, Size:** $\#S = (\Sigma x \mid x \in S : 1)$

(11.13) **Axiom, Subset:** $S \subseteq T \equiv (\forall x \mid x \in S : x \in T)$

(11.14) **Axiom, Proper subset:** $S \subset T \;\equiv\; S \subseteq T \land S \neq T$

(11.15) **Axiom, Superset:** $T \supseteq S \;\equiv\; S \subseteq T$

(11.16) **Axiom, Proper superset:** $T \supset S \;\equiv\; S \subset T$

(11.17) **Axiom, Complement:** $v \in \sim S \;\equiv\; v \in \mathbf{U} \land v \notin S$

(11.18) $v \in \sim S \;\equiv\; v \notin S$, for v in $\mathbf{U}$

(11.19) $\sim\sim S = S$

(11.20) **Axiom, Union:** $v \in S \cup T \;\equiv\; v \in S \lor v \in T$

(11.21) **Axiom, Intersection:** $v \in S \cap T \;\equiv\; v \in S \land v \in T$

(11.22) **Axiom, Difference:** $v \in S - T \;\equiv\; v \in S \land v \notin T$

(11.23) **Axiom, Power set:** $v \in \mathscr{P}S \;\equiv\; v \subseteq S$

(11.24) **Definition.** Let E_s be a set expression constructed from set variables, $\emptyset$, $\mathbf{U}$, $\sim$, $\cup$, and $\cap$.
Then E_p is the expression constructed from E_s by replacing:
$\emptyset$ with $false$, $\mathbf{U}$ with $true$, $\cup$ with $\lor$, $\cap$ with $\land$, $\sim$ with $\neg$.
The construction is reversible: E_s can be constructed from E_p.

(11.25) **Metatheorem.** For any set expressions E_s and F_s:
(a) $E_s = F_s$ is valid iff $E_p \equiv F_p$ is valid,
(b) $E_s \subseteq F_s$ is valid iff $E_p \Rightarrow F_p$ is valid,
(c) $E_s = \mathbf{U}$ is valid iff E_p is valid.

Basic properties of $\cup$.

(11.26) **Symmetry of $\cup$:** $S \cup T = T \cup S$

(11.27) **Associativity of $\cup$:** $(S \cup T) \cup U = S \cup (T \cup U)$

(11.28) **Idempotency of $\cup$:** $S \cup S = S$

(11.29) **Zero of $\cup$:** $S \cup \mathbf{U} = \mathbf{U}$

(11.30) **Identity of $\cup$:** $S \cup \emptyset = S$

(11.31) **Weakening:** $S \subseteq S \cup T$

(11.32) **Excluded middle:** $S \cup \sim S = \mathbf{U}$

Basic properties of $\cap$.

(11.33) **Symmetry of $\cap$:** $S \cap T = T \cap S$

(11.34) **Associativity of $\cap$:** $(S \cap T) \cap U = S \cap (T \cap U)$

(11.35) **Idempotency of $\cap$:** $S \cap S = S$

(11.36) **Zero of $\cap$:** $S \cap \emptyset = \emptyset$

(11.37) **Identity of $\cap$:** $S \cap \mathbf{U} = S$

(11.38) **Strengthening:** $S \cap T \subseteq S$

(11.39) **Contradiction:** $S \cap \sim S = \emptyset$

Basic properties of combinations of $\cup$ and $\cap$.

(11.40) **Distributivity of $\cup$ over $\cap$:** $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$

(11.41) **Distributivity of $\cap$ over $\cup$:** $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$

(11.42) **De Morgan:**

(a) $\sim (S \cup T) = \sim S \cap \sim T$

(b) $\sim (S \cap T) = \sim S \cup \sim T$

Additional properties of $\cup$ and $\cap$.

(11.43) $S \subseteq T \wedge U \subseteq V \Rightarrow (S \cup U) \subseteq (T \cup V)$

(11.44) $S \subseteq T \wedge U \subseteq V \Rightarrow (S \cap U) \subseteq (T \cap V)$

(11.45) $S \subseteq T \equiv S \cup T = T$

(11.46) $S \subseteq T \equiv S \cap T = S$

(11.47) $S \cup T = \mathbf{U} \equiv (\forall x \mid x \in \mathbf{U} : x \notin S \Rightarrow x \in T)$

(11.48) $S \cap T = \emptyset \equiv (\forall x \mid : x \in S \Rightarrow x \notin T)$

Properties of set difference.

(11.49) $S - T = S \cap \sim T$

(11.50) $S - T \subseteq S$

(11.51) $S - \emptyset = S$

(11.52) $S \cap (T - S) = \emptyset$

(11.53) $S \cup (T - S) = S \cup T$

(11.54) $S - (T \cup U) = (S - T) \cap (S - U)$

(11.55) $S - (T \cap U) = (S - T) \cup (S - U)$

Implication versus subset.

(11.56) $(\forall x \mid : P \Rightarrow Q) \equiv \{x \mid P\} \subseteq \{x \mid Q\}$

Properties of subset.

(11.57) **Antisymmetry:** $S \subseteq T \wedge T \subseteq S \equiv S = T$

(11.58) **Reflexivity:** $S \subseteq S$

(11.59) **Transitivity:** $S \subseteq T \wedge T \subseteq U \Rightarrow S \subseteq U$

(11.60) $\emptyset \subseteq S$

(11.61) $S \subset T \equiv S \subseteq T \wedge \neg(T \subseteq S)$

(11.62) $S \subset T \equiv S \subseteq T \wedge (\exists x \mid x \in T : x \notin S)$

(11.63) $S \subseteq T \equiv S \subset T \vee S = T$

(11.64) $S \not\subset S$

(11.65) $S \subset T \Rightarrow S \subseteq T$

(11.66) $S \subset T \Rightarrow T \not\subseteq S$

(11.67) $S \subseteq T \Rightarrow T \not\subset S$

(11.68) $S \subseteq T \wedge \neg(U \subseteq T) \Rightarrow \neg(U \subseteq S)$

(11.69) $(\exists x \mid x \in S : x \notin T) \Rightarrow S \neq T$

(11.70) **Transitivity:**

(a) $S \subseteq T \wedge T \subset U \Rightarrow S \subset U$

(b) $S \subset T \wedge T \subseteq U \Rightarrow S \subset U$

(c) $S \subset T \wedge T \subset U \Rightarrow S \subset U$

Theorems concerning power set $\mathscr{P}$.

(11.71) $\mathscr{P}\emptyset = \{\emptyset\}$

(11.72) $S \in \mathscr{P}S$

(11.73) $\#(\mathscr{P}S) = 2^{\#S}$ (for finite set S)

Union and intersection of families of sets.

(11.74.1) **Definition:** $v \in (\cup x \mid R : E) \equiv (\exists x \mid R : v \in E)$

(11.75.1) **Definition:** $v \in (\cap x \mid R : E) \equiv (\forall x \mid R : v \in E)$

(11.76) **Axiom, Partition:** Set S partitions T if

(i) the sets in S are pairwise disjoint and

(ii) the union of the sets in S is T, that is, if

$(\forall u, v \mid u \in S \wedge v \in S \wedge u \neq v : u \cap v = \emptyset) \wedge (\cup u \mid u \in S : u) = T$

Bags.

(11.79) **Axiom, Membership:** $v \in \{\!\![\, x \mid R : E \,]\!\!\} \equiv (\exists x \mid R : v = E)$

(11.80) **Axiom, Size:** $\#\{\!\![\, x \mid R : E \,]\!\!\} = (\Sigma x \mid R : 1)$

(11.81) **Axiom, Number of occurrences:** $v\#\{\!\![\, x \mid R : E \,]\!\!\} = (\Sigma x \mid R \wedge v = E : 1)$

(11.82) **Axiom, Bag equality:** $B = C \equiv (\forall v \mid : v\#B = v\#C)$

(11.83) **Axiom, Subbag:** $B \subseteq C \equiv (\forall v \mid : v\#B \leq v\#C)$

(11.84) **Axiom, Proper subbag:** $B \subset C \equiv B \subseteq C \wedge B \neq C$

(11.85) **Axiom, Union:** $B \cup C = \{\!\![\, v, i \mid 0 \leq i < v\#B + v\#C : v \,]\!\!\}$

(11.86) **Axiom, Intersection:** $B \cap C = \{\!\![\, v, i \mid 0 \leq i < v\#B \downarrow v\#C : v \,]\!\!\}$

(11.87) **Axiom, Difference:** $B - C = \{\!\![\, v, i \mid 0 \leq i < v\#B - v\#C : v \,]\!\!\}$

Mathematical Induction

(12.3) **Axiom, Mathematical Induction over $\mathbb{N}$:**

$(\forall n : \mathbb{N} \mid : (\forall i \mid 0 \leq i < n : P.i) \Rightarrow P.n) \Rightarrow (\forall n : \mathbb{N} \mid : P.n)$

(12.4) **Mathematical Induction over $\mathbb{N}$:**

$(\forall n : \mathbb{N} \mid : (\forall i \mid 0 \leq i < n : P.i) \Rightarrow P.n) \equiv (\forall n : \mathbb{N} \mid : P.n)$

(12.5) **Mathematical Induction over $\mathbb{N}$:**

$P.0 \wedge (\forall n : \mathbb{N} \mid : (\forall i \mid 0 \leq i \leq n : P.i) \Rightarrow P(n+1)) \equiv (\forall n : \mathbb{N} \mid : P.n)$

(12.11) **Definition, b to the power n:**

$b^0 = 1$

$b^{n+1} = b \cdot b^n$ for $n \geq 0$

(12.12) **b to the power n:**
$b^0 = 1$
$b^n = b \cdot b^{n-1}$ for $n \geq 1$

(12.13) **Definition, factorial:**
$0! = 1$
$n! = n \cdot (n-1)!$ for $n > 0$

(12.14) **Definition, Fibonacci:**
$F_0 = 0, \quad F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$ for $n > 1$

(12.14.1) **Definition, Golden Ratio:** $\phi = (1+\sqrt{5})/2 \approx 1.618 \qquad \hat{\phi} = (1-\sqrt{5})/2 \approx -0.618$

(12.15) $\phi^2 = \phi + 1$ and $\hat{\phi}^2 = \hat{\phi} + 1$

(12.16) $F_n \leq \phi^{n-1}$ for $n \geq 1$

(12.16.1) $\phi^{n-2} \leq F_n$ for $n \geq 1$

(12.17) $F_{n+m} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n$ for $n \geq 0$ and $m \geq 1$

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