Assignments
for
Math 220, Formal Methods

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Chapter, Section, and Exercise numbers in these assignments refer to the text for this course, *A Logical Approach to Discrete Math*, David Gries and Fred B. Schneider, Springer-Verlag, 1994, ISBN 0-387-94115-0. Exercise numbers have no parentheses, as in "Do Exercises 1.1, 1.2, 1.3." Theorem numbers have parentheses, as in "Prove (3.12) Double negation."

1. Read lightly Chapter 0.
2. Study Sections 1.1, 1.2.
3. Do Exercises 1.1, 1.2, 1.3.

Submission requirements

All homework must be submitted electronically as a single `.pdf` file to the Assignments page on the Courses website for this course. The name of the file must begin with a lowercase `a`, followed by a two-digit assignment number, followed by your last name in all lowercase. For example, if your last name is Smith then the name of homework file for this first assignment must be

```
a01smith.pdf
```

For the second assignment, the name of the file must be

```
a02smith.pdf
```

and so on.

There are several ways to create the `.pdf` file. You may write your solutions on a mobile device like an iPad, which will create the `.pdf` file directly. Or, you may write your solutions on paper then scan your written work into a `.pdf` document. IMPORTANT: If you do your work on paper you MUST use pencil and eraser. Pens are not allowed.

Multiple files for a single assignment are not permitted. If your work fills more than one page, you must not hand in a separate file for each page. Instead, they must be combined into a single multi-page `.pdf` file. You must not simply photograph your paper. If you do not have access to a scanner, use a scanner app that has the capability of scanning directly into a multi-page `.pdf` document.

Do all your work on a single column. Do not use a second column in the middle of the page. The grader needs the space in the middle of the page to comment on your solutions. The grader will return your work with pdf markup in the Courses Drop Box.

1. Study Sections 1.3, 1.4, 1.5.
2. Do Exercises 1.7, 1.8, 1.9.

1. Study Sections 2.1, 2.2.
2. Do Exercises 1.11, 2.1(a, b, c, d, g, j), 2.2(d, e, h).

1. Study Sections 2.3, 2.4.

2. Study English expressions handout.

3. Do Exercises 2.3(d, e, h), 2.4(b, d, f).

4. Do Exercise 2.5. Use

   r for "It's raining,"
   s for "I'm going swimming,"
   sc for "It's raining cats,"
   sd for "It's raining dogs,"
   eh for "I'll eat my hat."

5. Do Exercise 2.7. Use

   xly for $x < y$,
   xey for $x = y$,
   xgy for $x > y$,
   ylz for $y < z$,
   ygz for $y > z$,
   vew for $v = w$,
   $\neg xly$ for $x \geq y$,
   ep for "Execution of P is begun with $x < y$,"
   ty for "Execution of P terminates with $y = 2x$,"
   $ep1$ for "Execution of P is begun with $x < 0$,"
   ept for "Execution of P terminates."

   Hint for part (e): None or one of the three must be true. So, the expression should start like this. $(\neg xly \wedge \neg ylz \wedge \neg vew) \vee (xly \wedge \neg ylz \wedge \neg vew) \vee \ldots, etc.$

   Hint for part (i): The word "means" translates to $\equiv$.

6. Translate the following English sentences into boolean expressions. Use

   xgy for $x > y$,
   ylz for $y < z$.

   (a) $x > y$ if $y < z$.
   (b) $x > y$ iff $y < z$.
   (c) $x > y$ only if $y < z$.
   (d) $x > y$ if and only if $y < z$.
   (e) $x > y$ is a sufficient condition for $y < z$.
   (f) $x > y$ is a necessary condition for $y < z$.
   (g) $x > y$ is a necessary and sufficient condition for $y < z$.
   (h) $x > y$ whenever $y < z$.
   (i) $x > y$ provided that $y < z$.
   (j) $x > y$ unless $y < z$.
   (k) $x > y$ unless it is not the case that $y < z$.

1. Study Sections 3.1, 3.2, 3.3.
2. Prove (3.12) Double negation.
3. Prove (3.13) Negation of false.
4. Prove (3.14).
5. Prove (3.19) Mutual interchageability.
   See the hint in Exercise 3.14.

1. Study Sections 3.4, 3.5.
2. Prove (3.30) Identity of $\vee$.
   See the hint in Exercise 3.17.
3. Prove (3.31) Distributivity of $\vee$ over $\vee$.
   See the hint in Exercise 3.18.
4. Prove (3.32).
5. Prove (3.36) Symmetry of $\wedge$.
   See the hint in Exercise 3.22.
6. Prove (3.38) Idempotency of $\wedge$.
   See the hint in Exercise 3.23.
7. Prove (3.47a) De Morgan.
   The trick is to use (3.32) twice. You can do the proof in four steps starting with the LHS. Here are the hints for the first three steps.

   $\langle$Golden Rule$\rangle$
   $\langle$(3.9) with the $\neg$ applied to just the first $p\rangle$
   $\langle$(3.32) with $p, q := q, p\rangle$
8. Prove (3.47b) De Morgan.
   Start with $\neg(p \vee q)$ and pull two rabbits out of a hat by using (3.12) double negation on the p and on the q. Then use (3.47a) De Morgan.

To prove theorems with implication, the simplest approach is usually to use (3.59).

1. Study Section 3.6 except Leibniz's Rule as an Axiom.
2. Prove (3.61) Contrapositive.
3. Prove (3.63) Distributivity of $\Rightarrow$ over $\equiv$.
4. Prove (3.65) Shunting.
   See the hint in Exercise 3.49.
5. Prove (3.70).
   See the hint in Exercise 3.54.
6. Prove (3.71) Reflexivity of $\Rightarrow$.
7. Prove (3.72) Right zero of $\Rightarrow$.
8. Prove (3.76a) Weakening the consequent.
9. Prove (3.77) Modus ponens.
   Use theorem (3.66).

1. Study Section 3.6, Leibniz's Rule as an Axiom, 4.1.
2. Prove (3.85a) Replace by *true*.
   See the hint in Exercise 3.76.
3. Prove (3.85b) Replace by *true*.
4. Prove (4.1) using the method of Section 4.1.
   That is, start with $(q \Rightarrow p)$ and get it to follow from p. This method is a requirement, not a hint.
5. Prove (4.3) Monotonicity of $\wedge$.
   Start with the RHS, eliminate the $\Rightarrow$ with (3.59) followed by (3.47a) De Morgan and (3.45) Distributivity of $\vee$ over $\wedge$. After some simplifications you can use (3.76a) Strengthening to introduce $\Leftarrow$.

1. Study Section 4.2.
2. Prove (4.1) using the method of assuming the conjunct of the antecedent, p.
3. Re-prove (3.77) Modus ponens using the method of assuming the conjuncts of the antecedent, p and $p \Rightarrow q$. Start with the consequent q, and use (3.57) in your first step.

1. Study Section 5.1.
2. Do Exercise 5.1(b).
3. Re-prove (3.76e) Weakening/strengthening using the method of assuming the conjuncts of the antecedent.
4. Re-prove (3.46) Distributivity of $\wedge$ over $\vee$ using case analysis on p.

1. Do Exercise 5.1(a, c, d).
2. Re-prove (3.47a) De Morgan using proof by contradiction.
   For this problem, you must start with (3.9) as the first step. You are not allowed to simply replicate the proof for (3.47a) and then negate the expression at the end of the proof to make it false.
3. Re-prove (3.76c) Weakening/strengthening using proof by contrapositive.

1. Study Section 8.1.
2. Do Exercise 5.6(a).
3. Do Exercise 5.6(b).
   Remove the implication using (3.57), then use (3.11) to move the negation to the disjunction term.
4. Do Exercise 5.6(c).
5. Do Exercise 8.1.

1. Study Section 8.2.
2. Do Exercise 8.3.

1. Study Section 8.3.
2. Practice proving (8.22), Change of dummy, which will be on Test 2.

1. Study Section 8.4.
2. Do Exercise 8.5.
   These are all one-step proofs!
3. Prove (8.23b), Split off term.
4. Extra credit. Not required.
   The proof of (8.23a) Split off term uses the fact that

   $0 \leq i < n+1 \;\equiv\; 0 \leq i < n \vee i = n$

   under the assumption that n: $\mathbb{N}$, that is, $0 \leq n$. Give a detailed proof of this fact by filling in the steps of the formal proof below from the hints.

$0 \leq i < n+1$
$=$ ⟨Remove the conjunctive abbreviation⟩

$=$ $\langle i < n+1 \;\equiv\; i < n \vee i = n \rangle$

$=$ ⟨(3.46) $\wedge$ distributes over $\vee$⟩

$=$ ⟨(3.84a) Substitution⟩

$=$ ⟨Assumption $0 \leq n$⟩

$=$ ⟨(3.39) Identity of $\wedge$⟩

$=$ ⟨Reintroduce conjunctive abbreviation⟩
$0 \leq i < n \vee i = n$

1. Study Section 9.1.

2. Prove (9.6).
   See the hint in Exercise 9.3.

3. Prove (9.8).
   See the hint in Exercise 9.4.

4. Prove (9.9).
   See the hint in Exercise 9.5.

5. Prove (9.11) Body weakening/strengthening.
   See the hint in Exercise 9.8.

6. Prove (9.13) Instantiation: $(\forall x \mid : P) \Rightarrow P[x := E]$. Note that instantiation involves a textual substitution on the right side of an implication. The only previous axiom or theorem that relates a quantified expression with a textual substitution is the one-point rule (8.14). The problem is that the one-point rule has the restriction that x does not occur in E, while instantiation has no such restriction. So, to prove instantiation without the restriction you must first use dummy renaming (8.21), which also uses the *occurs* predicate. The proof is outlined below with the hints supplied and the proof steps for you to fill in.

$(\forall x \mid : P)$

$=$ $\langle$(8.21) Dummy renaming with $y, R := z, true$, and z chosen to not occur in $E\rangle$

$=$ $\langle$(3.28) Excluded middle with $p := z = E\rangle$

$\Rightarrow$ $\langle$(9.10) Range weakening$\rangle$

$=$ $\langle$(8.14) One-point rule, which is legal now because z is not free in $E\rangle$

$=$ $\langle$Property of textual substitution, because z is not free in $P\rangle$

1. Study Section 9.2.
2. Prove (9.18c) Generalized De Morgan.
3. Prove (9.19) Trading.
4. Prove (9.20) Trading.
5. Prove (9.22).
6. Prove (9.26) Body weakening/strengthening.
7. Prove (9.28) Existential-introduction.

1. Study Section 9.3.
2. Do Exercise 9.29 all except (g) and (m).
   Do not use any predicates. Instead, write quantified expressions.
3. Do Exercise 9.34.
   Define $\text{loves}(x, y)$: Person x loves person y.
4. Do Exercise 9.35.
   Define $\text{fool}(p, t)$: You can fool person p at time t.

1. Study Sections 11.1, 11.2.

2. Do Exercise 11.1(a, b, e, f, g, h).
   Include the type of the dummy in the set comprehension. Do not use any predicates except for prime.i. If you expand any predicate into a quantified expression, you may use only addition, multiplication, and exponentiation in the quantified expression.

3. Do Exercise 11.2(a, b, c, d).

4. Do Exercise 11.3.
   Some early printings of the book have $\equiv$ between the two set expressions. It should be $=$.

1. Study Sections 11.3, 11.4.
2. Read lightly Sections 11.6, 11.7.
3. Do Exercise 11.6.
4. Do Exercise 11.7(a).
   Do not assume part (d). Starting with the LHS of the $\equiv$ begin with the following steps in this order: (11.2), (11.5.1), (11.9), (3.80), (8.15).
5. Prove (11.4.2).
6. Prove (11.7).
   First write $\{x \mid R\}$ in nonabbreviated form, then rename the dummy variable x to y using (8.21) before applying (11.3).

1. Do Exercise 11.14.

2. Prove (11.45).
   You cannot use metatheorem (11.25) because you are proving an equivales between two boolean expressions, not a set equality.

3. Prove (11.46).
   You cannot use metatheorem (11.25) because you are proving an equivales between two boolean expressions, not a set equality.

4. Prove (11.48).
   You cannot use metatheorem (11.25) because you are proving an equivales between two boolean expressions, not a set equality.

5. Prove (11.49).
   You cannot use the metatheorem. Use the technique of (11.11b). Some early printings of the book have equivales between the two set expressions. It should be equals.

6. Prove (11.50).
   Manipulate to get a form that will allow you to use the metatheorem.

7. Prove (11.57).
   Start with (11.13) twice on the LHS.

8. Prove (11.58).

1. Study Section 12.1.
2. Do Exercise 12.4(a).
3. Prove (11.60).
4. Prove (11.61).
   Start with the LHS and use (11.14) and (11.57) and some propositional calculus.
5. Prove (11.72).