For symmetric and associative binary operator $\star$ with identity $u$.

(8.13)     **Axiom, Empty range:**    $(\star x \mid false : P) = u$

$0$ is the identity of $+$.

$(+i \mid 2 \leq i < 5 : i^2) = 2^2 + 3^2 + 4^2$

$(+i \mid 2 \leq i < 4 : i^2) = 2^2 + 3^2$

$(+i \mid 2 \leq i < 3 : i^2) = 2^2$

$(+i \mid 2 \leq i < 2 : i^2) = (+i \mid false : i^2) = 0$

$true$ is the identity of $\wedge$.

Suppose $b$ is an array of integers.

$(\wedge i \mid 2 \leq i < 5 : b[i] < x) = b[2] < x \wedge b[3] < x \wedge b[4] < x$

$(\wedge i \mid 2 \leq i < 4 : b[i] < x) = b[2] < x \wedge b[3] < x$

$(\wedge i \mid 2 \leq i < 3 : b[i] < x) = b[2] < x$

$(\wedge i \mid 2 \leq i < 2 : b[i] < x) = (\wedge i \mid false : b[i] < x) = true$

(8.14)     **Axiom, One-point rule:**   Provided $\neg occurs('x', 'E')$,

        $(\star x \mid x = E : P) = P[x := E]$

$(+i \mid i = 3 : i^2) = i^2 [i := 3] = 3^2$

Suppose $b$ is an array of integers.

$(\vee i \mid i = 3 : b[i] < x) = (b[i] < x)[i := 3] = b[3] < x$

(8.15)     **Axiom, Distributivity:**   Provided $P, Q : \mathbb{B}$ or $R$ is finite,

        $(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

    $(+i \mid 1 \leq i < 4 : 2i) + (+i \mid 1 \leq i < 4 : 5i^2)$

$=$   $\langle$Expand quantifications$\rangle$

    $(2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3) + (5 \cdot 1^2 + 5 \cdot 2^2 + 5 \cdot 3^2)$

$=$   $\langle$Symmetry and associativity of $+\rangle$

    $(2 \cdot 1 + 5 \cdot 1^2) + (2 \cdot 2 + 5 \cdot 2^2) + (2 \cdot 3 + 5 \cdot 3^2)$

$=$   $\langle$Quantify$\rangle$

    $(+i \mid 1 \leq i < 4 : 2i + 5i^2)$

(8.16)    **Axiom, Range split:**    Provided $R \wedge S \equiv false$ and $P : \mathbb{B}$ or $R$ and $S$ are finite,
$$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$$

$R : 0 \le i < 3$
$S : 6 \le i < 9$
$R \vee S : 0 \le i < 3 \ \vee \ 6 \le i < 9$
$R \wedge S : false$

$\quad (+i \mid R \vee S : i^2)$
$= \quad \langle \text{Definition of } R \text{ and } S \rangle$
$\quad (+i \mid 0 \le i < 3 \ \vee \ 6 \le i < 9 : i^2)$
$= \quad \langle \text{Expand quantification} \rangle$
$\quad 0^2 + 1^2 + 2^2 + 6^2 + 7^2 + 8^2$
$= \quad \langle \text{Associativity of } + \rangle$
$\quad (0^2 + 1^2 + 2^2) + (6^2 + 7^2 + 8^2)$
$= \quad \langle \text{Quantify} \rangle$
$\quad (+i \mid 0 \le i < 3 : i^2) + (+i \mid 6 \le i < 9 : i^2)$
$= \quad \langle \text{Definition of } R \text{ and } S \rangle$
$\quad (+i \mid R : i^2) + (+i \mid S : i^2)$

(8.17)    **Axiom, Range split:**    Provided $P : \mathbb{B}$ or $R$ and $S$ are finite,
$$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$$

Now, $R \wedge S$ is not required to be false.

$R : 1 \le i < 5$
$S : 3 \le i < 7$
$R \vee S : 1 \le i < 7$
$R \wedge S : 3 \le i < 5$

$\quad (+i \mid R \vee S : i^2) + (+i \mid R \wedge S : i^2)$
$= \quad \langle \text{Definition of } R \text{ and } S \rangle$
$\quad (+i \mid 1 \le i < 7 : i^2) + (+i \mid 3 \le i < 5 : i^2)$
$= \quad \langle \text{Expand quantifications} \rangle$
$\quad (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) + (3^2 + 4^2)$
$= \quad \langle \text{Symmetry and associativity of } + \rangle$
$\quad (1^2 + 2^2 + 3^2 + 4^2) + (3^2 + 4^2 + 5^2 + 6^2)$
$= \quad \langle \text{Quantify} \rangle$
$\quad (+i \mid 1 \le i < 5 : i^2) + (+i \mid 4 \le i < 5 : i^2)$
$= \quad \langle \text{Definition of } R \text{ and } S \rangle$
$\quad (+i \mid R : i^2) + (+i \mid S : i^2)$

(8.18)    **Axiom, Range split for idempotent $\star$ :**    Provided $P : \mathbb{B}$ or $R$ and $S$ are finite,
$$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$$

$\wedge$ is idempotent because $p \wedge p \equiv p$.
Suppose $b$ is an array of integers.
$R : 0 \leq i < 2$
$S : 1 \leq i < 3$
$R \vee S : 0 \leq i < 3$

$\quad (\wedge i \mid R : x < b[i]) \wedge (\wedge i \mid S : x < b[i])$
$= \quad \langle$Definition of $R$ and $S\rangle$
$\quad (\wedge i \mid 0 \leq i < 2 : x < b[i]) \wedge (\wedge i \mid 1 \leq i < 3 : x < b[i])$
$= \quad \langle$Expand quantifications$\rangle$
$\quad x < b[0] \wedge x < b[1] \wedge x < b[1] \wedge x < b[2]$
$= \quad \langle(3.38)\ p \wedge p \equiv p\rangle$
$\quad x < b[0] \wedge x < b[1] \wedge x < b[2]$
$= \quad \langle$Quantify$\rangle$
$\quad (\wedge i \mid 0 \leq i < 3 : x < b[i])$
$= \quad \langle$Definition of $R$ and $S\rangle$
$\quad (\wedge i \mid R \vee S : x < b[i])$

(8.19)    **Axiom, Interchange of dummies:**    Provided $\star$ is idempotent or $R$ and $Q$ are finite,
$\quad \neg occurs('y','R'), \neg occurs('x','Q'),$
$$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$$

$R : 1 \leq x < 4$
$Q : 8 \leq y < 10$
$P : 6 \cdot x \cdot y$

Note that $\neg occurs('y','1 \leq x < 4')$ and $\neg occurs('x','8 \leq y < 10')$

$\quad (+x \mid R : (+y \mid Q : 6 \cdot x \cdot y))$
$= \quad \langle$Expand inner quantification$\rangle$
$\quad (+x \mid R : 6 \cdot x \cdot 8 + 6 \cdot x \cdot 9)$
$= \quad \langle$Expand quantification$\rangle$
$\quad (6 \cdot 1 \cdot 8 + 6 \cdot 1 \cdot 9) + (6 \cdot 2 \cdot 8 + 6 \cdot 2 \cdot 9) + (6 \cdot 3 \cdot 8 + 6 \cdot 3 \cdot 9)$
$= \quad \langle$Symmetry and associativity of $+\rangle$
$\quad (6 \cdot 1 \cdot 8 + 6 \cdot 2 \cdot 8 + 6 \cdot 3 \cdot 8) + (6 \cdot 1 \cdot 9 + 6 \cdot 2 \cdot 9 + 6 \cdot 3 \cdot 9)$
$= \quad \langle$Quantify over $x\rangle$
$\quad (+x \mid R : 6 \cdot x \cdot 8) + (+x \mid R : 6 \cdot x \cdot 9)$
$= \quad \langle$Quantify over $y\rangle$
$\quad (+y \mid Q : (+x \mid R : 6 \cdot x \cdot y))$